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Deriving interval weights from an interval multiplicative consistent fuzzy preference relation

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ABSTRACT

In this paper, relations between a multiplicative consistent interval fuzzy preference relation and an additive consistent interval fuzzy preference relation are established. Based on the new relations, a new method (Algorithm 3) is proposed to derive interval weights by transforming a multiplicative consistent interval fuzzy preference relation into an additive consistent interval fuzzy preference relation, collecting additive consistent information (Algorithm 2), transforming back into multiplicative consistent information and calculating the interval weights by Eq. (20). Finally, two numerical examples are given to illustrate the new method.

1. Introduction

Multiple attribute decision making (MADM) is used to deal with the problems of finding a desirable solution from a finite set of feasible alternatives assessed on multiple attributes [1–7]. One key of MADM is deriving the attributes' weights which can be divided into subjective weights, objective weights and combination weights. In this paper, subjective weights which reflect decision makers (DMs)' preferences will be discussed. As it is known, both fuzzy preference relation (FPR) and reciprocal preference relation (RPR) [8,9] are used to express decision makers preferences. Since the analytic hierarchy process (AHP) was proposed by Saaty in the middle of the 1970s, deriving weights from DMs' preferences has attracted researchers' interest. For example, Saaty [10] firstly proposed the well-known eigenvector method (EM) to derive weights from a multiplicative preference relation. Then, logarithmic least-squares method (LLSM) [11], gradient eigenweight method (GEM) [12], geometric least-squares method (GLSM) [13] and logarithmic goal programming method (LGPM) [14] were developed for a multiplicative preference relation. Besides, to derive weights from a FPR, Fernandez and Leyva [15] proposed a multi-objective optimization method, Xu [16] developed a goal programming model, and so on.

However, DMs may not exactly estimate their preferences with numerical values, it is natural and easy for DMs expressing their preference information with interval numbers due to the increasing complexity and uncertainty of real-life decision making problems [17]. In this case, both interval fuzzy preference relation (IFPR) and interval reciprocal preference relation (IRPR) are useful to express DMs' uncertain preferences. So, deriving interval weights from interval preference relations is the key of multiple attribute decision making problems. Up to now, Lan et al. [18] proposed an information mining method to derive weights from an interval comparison matrix. Wang et al. [19] developed an approach generating interval weights based on consistency test. Xu and Chen [17] established some models for deriving interval weights from IFPR. Genc et al. [20] proposed a new method by adjusting elements of the IFPR. In short, almost all the research obtain interval weights by establishing mathematical models. This paper is focused on deriving interval weights by collecting all the multiplicative consistent information. To do that, some exchanges between an additive consistent FPR and a multiplicative consistent FPR are established based on both relations between a consistent reciprocal preference and a multiplicative consistent FPR proposed by Xu [3] and relations between a consistent reciprocal preference and an additive consistent FPR proposed by Lan [4]. Then, the relations are extended to a multiplicative consistent IFPR and an additive consistent IFPR. Based on the extended relations, a new method (Algorithm 3) will be proposed to derive interval weights from an IFPR, no matter whether it is multiplicative consistent or not, by transforming a multiplicative consistent IFPR into an additive consistent IFPR, collecting additive consistent information (Algorithm 2), transforming back into multiplicative consistent

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information and calculating the interval weights by Eq. (20). Finally, there are two numerical examples to illustrate the new method.

2. Preliminaries

Consider a certain multiple criteria decision making problem with a finite set of n criteria, let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of criteria and let $I = \{1, 2, \ldots, n\}$ be the set of index. A decision maker compares each pair of criteria in X, and provides his/her preference degree a_{ij} of the criterion x_i over x_j. All these preference degrees a_{ij} $(i,j \in I)$ compose a FPR $A = (a_{ij})_{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Definition 1 [21]. A fuzzy preference relation on a finite set X with $n \geqslant 1$ elements is represented by a complementary matrix $A = (a_{ij})_{n\times n}$ with

$$a_{ij} \geqslant 0, \quad a_{ij} + a_{ji} = 1, \quad a_{ii} = 0.5, \quad \forall i,j \in I, \tag{1}$$

where a_{ij} represents a crisp preference degree of criterion x_i over x_j provided by the decision maker. Especially, $a_{ij} < 0.5$ indicates that x_j is preferred to x_i; $a_{ij} = 0.5$ indicates indifference between x_i and x_j; $a_{ij} > 0.5$ indicates that x_i is preferred to x_j.

Definition 2 [1]. A reciprocal preference relation R on a finite set X with $n \geqslant 1$ elements is represented by a reciprocal matrix $R = (r_{ij})_{n\times n}$ with

$$r_{ij} > 0, \quad r_{ij} \cdot r_{ji} = 1, \quad r_{ii} = 1, \quad \forall i,j \in I, \tag{2}$$

where r_{ij} indicates that x_i is r_{ij} times as important as x_j. Especially, $r_{ij} < 1$ indicates that x_j is preferred to x_i; $r_{ij} = 1$ indicates indifference between x_i and x_j; $r_{ij} > 1$ indicates that x_i is preferred to x_j.

Definition 3 [22]. A fuzzy preference relation $A = (a_{ij})_{n\times n}$ is called an additive consistent fuzzy preference relation, if the following additive transitivity is satisfied

$$a_{ij} = a_{ik} - a_{jk} + 0.5, \quad \forall i,j,k \in I. \tag{3}$$

Definition 4 [22]. A fuzzy preference relation $B = (b_{ij})_{n\times n}$ is called a multiplicative consistent fuzzy preference relation, if the following transitivity is fulfilled

$$b_{ij} \cdot b_{jk} \cdot b_{ki} = b_{ik} \cdot b_{kj} \cdot b_{ji}, \quad \forall i,j,k \in I. \tag{4}$$

Let $w = (w_1, w_2, \ldots, w_n)^T$ be the vector of priority weights, with $w_i \geqslant 0$ $(i \in I)$, $\sum_{i=1}^n w_i = 1$. Then the multiplicative consistent fuzzy preference relation B can be given by Xu [2]

$$b_{ij} = \frac{w_i}{w_i + w_j}, \quad \forall i,j \in I. \tag{5}$$

Definition 5 [23]. A reciprocal preference relation $R = (r_{ij})_{n\times n}$ is called a consistent reciprocal preference relation, if the following transitivity is satisfied

$$r_{ij} = r_{ik} \cdot r_{kj}, \quad \forall i,j,k \in I. \tag{6}$$

Lemma 1 [3]. *Let $R = (r_{ij})_{n\times n}$ be a consistent reciprocal preference relation, a multiplicative consistent fuzzy preference relation $B = (b_{ij})_{n\times n}$ will be generated through the following transformation*

$$b_{ij} = \frac{1}{1 + r_{ji}}, \quad \forall i,j \in I. \tag{7}$$

Lemma 2 [3]. *Let $B = (b_{ij})_{n\times n}$ be a multiplicative consistent fuzzy preference relation, a consistent reciprocal preference relation $R = (r_{ij})_{n\times n}$ will be generated through the following transformation*

$$r_{ij} = \frac{b_{ij}}{b_{ji}}, \quad \forall i,j \in I. \tag{8}$$

Lemma 3 [4]. *Let $R = (r_{ij})_{n\times n}$ be a consistent reciprocal preference relation, an additive consistent fuzzy preference relation $A = (a_{ij})_{n\times n}$ will be generated through the following transformation*

$$a_{ij} = 0.5 + \log_\alpha r_{ij}, \quad \forall i,j \in I, \ \alpha > \left(\max_{i,j\in I} r_{ij}\right)^2. \tag{9}$$

Lemma 4 [4]. *Let $A = (a_{ij})_{n\times n}$ be an additive consistent fuzzy preference relation, a consistent reciprocal preference relation $R = (r_{ij})_{n\times n}$ will be generated through the following transformation*

$$r_{ij} = \beta^{a_{ij} - 0.5}, \quad \forall i,j \in I, \ \beta > 1. \tag{10}$$

Lemmas 1 and 2 reflect the relations between a consistent reciprocal preference relation and a multiplicative consistent fuzzy preference relation and Lemmas 3 and 4 reflect the relations between a consistent reciprocal preference relation and an additive consistent fuzzy preference relation. The relations, between an additive consistent fuzzy preference relation and a multiplicative consistent fuzzy preference relation, can be easily achieved as follows.

Theorem 1. *Let $A = (a_{ij})_{n\times n}$ be an additive consistent fuzzy preference relation, a multiplicative consistent fuzzy preference relation $B = (b_{ij})_{n\times n}$ can be achieved through the following transformation*

$$b_{ij} = \frac{1}{1 + \beta^{a_{ji} - 0.5}}, \quad \forall i,j \in I, \beta > 1. \tag{11}$$

Proof. Since $A = (a_{ij})_{n\times n}$ is an additive consistent fuzzy preference relation, according to Lemma 4, $R = (r_{ij})_{n\times n}$ is a consistent reciprocal preference relation, where $r_{ij} = \beta^{a_{ij}-0.5}$ $(\forall i,j \in I, \ \beta > 1)$. According to Lemma 1, $B = (b_{ij})_{n\times n}$ is a multiplicative consistent fuzzy preference relation, where

$$b_{ij} = \frac{1}{1 + r_{ji}} = \frac{1}{1 + \beta^{a_{ji} - 0.5}}, \quad \forall i,j \in I, \ \beta > 1. \qquad \square$$

Theorem 2. *Let $B = (b_{ij})_{n\times n}$ be a multiplicative consistent fuzzy preference relation, an additive consistent fuzzy preference relation $A = (a_{ij})_{n\times n}$ can be achieved through the following transformation*

$$a_{ij} = 0.5 + \log_\alpha \frac{b_{ij}}{b_{ji}}, \quad \forall i,j \in I, \ \alpha > \left(\max_{i,j\in I} \frac{b_{ij}}{b_{ji}}\right)^2. \tag{12}$$

Proof. Since $B = (b_{ij})_{n\times n}$ is a multiplicative consistent fuzzy preference relation, according to Lemma 2, $R = (r_{ij})_{n\times n}$ is a consistent reciprocal preference relation, where $r_{ij} = \frac{b_{ij}}{b_{ji}}$ $(\forall i,j \in I)$. According to Lemma 3, $A = (a_{ij})_{n\times n}$ is an additive consistent fuzzy preference relation, where

$$a_{ij} = 0.5 + \log_\alpha r_{ij} = 0.5 + \log_\alpha \frac{b_{ij}}{b_{ji}}, \quad \forall i,j \in I, \ \alpha > \left(\max_{i,j\in I} \frac{b_{ij}}{b_{ji}}\right)^2. \qquad \square$$

From the above two theorems, we can see that, for an additive consistent fuzzy preference relation $A = (a_{ij})_{n\times n}$ and a multiplicative consistent fuzzy preference relation $B=(b_{ij})_{n\times n}$, when $\alpha=\beta$, Eq. (11) is the sufficient and necessary condition of Eq. (12).

As the increasing of the complexity and uncertainty of real-life, DMs can hardly estimate their preferences with numerical values, but with fuzzy set, especially with interval numbers. From this point, a decision maker compares each pair of criteria in X, and provides his/her interval preference degree $\tilde{a}_{ij}=\left[a_{ij}^-,a_{ij}^+\right]$ of the criterion x_i over x_j, where $\tilde{a}_{ij}$ indicates that the criterion x_i is between a_{ij}^- and a_{ij}^+ times as important as the criterion x_j. All these interval preference degrees $\tilde{a}_{ij}$ $(i,j\in I)$ compose an IFPR $\widetilde{A}=(\tilde{a}_{ij})_{n\times n}$.

$$\widetilde{A}=\begin{bmatrix}[a_{11}^-,a_{11}^+] & [a_{12}^-,a_{12}^+] & \cdots & [a_{1n}^-,a_{1n}^+]\\ [a_{21}^-,a_{21}^+] & [a_{22}^-,a_{22}^+] & \cdots & [a_{2n}^-,a_{2n}^+]\\ \vdots & \vdots & \vdots & \vdots\\ [a_{n1}^-,a_{n1}^+] & [a_{n2}^-,a_{n2}^+] & \cdots & [a_{nn}^-,a_{nn}^+]\end{bmatrix},$$

where $\tilde{a}_{ij}=\left[a_{ij}^-,a_{ij}^+\right],\ a_{ij}^-+a_{ji}^+=1,\ a_{ii}^-=a_{ii}^+=0.5,\ a_{ij}^+\geqslant a_{ij}^->0,$ for all $i,j=1,2,\ldots,n$.

Definition 6 [17]. Let $\widetilde{A}=(\left[a_{ij}^-,a_{ij}^+\right])_{n\times n}$ be an IFPR. If there exists an additive consistent FPR $A=(a_{ij})_{n\times n}$, such that

$$a_{ij}^-\leqslant a_{ij}\leqslant a_{ij}^+,\quad \forall i,j\in I,$$

then, $\widetilde{A}$ is called an additive consistent IFPR and $A=(a_{ij})_{n\times n}$ is called additive consistent information in $\widetilde{A}$.

Definition 7 [17]. Let $\widetilde{B}=\left(\left[b_{ij}^-,b_{ij}^+\right]\right)_{n\times n}$ be an IFPR. If there exists a multiplicative consistent FPR $B=(b_{ij})_{n\times n}$, such that

$$b_{ij}^-\leqslant b_{ij}\leqslant b_{ij}^+,\quad \forall i,j\in I,$$

then, $\widetilde{B}$ is called a multiplicative consistent IFPR and $B=(b_{ij})_{n\times n}$ is called multiplicative consistent information in $\widetilde{B}$.

The relationship between an additive consistent IFPR and multiplicative consistent IFPR can be developed as follows.

Theorem 3. *Let $\widetilde{A}=\left(\left[a_{ij}^-,a_{ij}^+\right]\right)_{n\times n}$ be an additive consistent IFPR. A multiplicative consistent IFPR $\widetilde{B}=\left(\left[b_{ij}^-,b_{ij}^+\right]\right)_{n\times n}$ can be achieved through the following transformation*

$$b_{ij}^-=\frac{1}{1+\beta^{a_{ji}^+-0.5}},\quad b_{ij}^+=\frac{1}{1+\beta^{a_{ji}^--0.5}},\quad \forall i,j\in I,\ \beta>1. \tag{13}$$

Proof. Since $\widetilde{A}=([a_{ij}^-,a_{ij}^+])_{n\times n}$ is an additive consistent IFPR, according to Definition 6, there exists an additive consistent FPR $A=(a_{ij})_{n\times n}$, such that $a_{ij}^-\leqslant a_{ij}\leqslant a_{ij}^+$ $(\forall i,j\in I)$. Further more, $\frac{1}{1+\beta^{a_{ji}^+-0.5}}\leqslant\frac{1}{1+\beta^{a_{ji}-0.5}}\leqslant\frac{1}{1+\beta^{a_{ji}^--0.5}}$ $(\forall i,j\in I,\beta>1)$. Besides, according to Theorem 1, $B=(b_{ij})_{n\times n}$ is a multiplicative consistent fuzzy preference relation, where $b_{ij}=\frac{1}{1+\beta^{a_{ji}-0.5}}$ $(\forall i,j\in I,\ \beta>1)$. So there exists a multiplicative consistent fuzzy preference relation $B=(b_{ij})_{n\times n}$, such that $b_{ij}^-\leqslant b_{ij}\leqslant b_{ij}^+$ $(\forall i,j\in I)$. According to Definition 7, $\widetilde{B}$ is a multiplicative consistent IFPR. □

Theorem 4. *Let $\widetilde{B}=\left(\left[b_{ij}^-,b_{ij}^+\right]\right)_{n\times n}$ be a multiplicative consistent IFPR. An additive consistent IFPR $\widetilde{A}=\left(\left[a_{ij}^-,a_{ij}^+\right]\right)_{n\times n}$ can be achieved through the following transformation*

$$a_{ij}^-=0.5+\log_\alpha\frac{b_{ij}^-}{b_{ji}^+},\quad a_{ij}^+=0.5+\log_\alpha\frac{b_{ij}^+}{b_{ji}^-},\quad \forall i,j\in I,\ \alpha>\left(\max_{i,j\in I}\frac{b_{ij}^+}{b_{ji}^-}\right)^2. \tag{14}$$

Proof. Since $\widetilde{B}=\left(\left[b_{ij}^-,b_{ij}^+\right]\right)_{n\times n}$ is a multiplicative consistent IFPR, according to Definition 7, there exists a multiplicative consistent FPR $B=(b_{ij})_{n\times n}$, such that $b_{ij}^-\leqslant b_{ij}\leqslant b_{ij}^+$ $(\forall i,j\in I)$. Further more, $\alpha>\left(\max_{i,j\in I}\frac{b_{ij}^+}{b_{ji}^-}\right)^2\geqslant 1$, then $0.5+\log_\alpha\frac{b_{ij}^-}{b_{ji}^+}\leqslant 0.5+\log_\alpha\frac{b_{ij}}{b_{ji}}\leqslant 0.5+\log_\alpha\frac{b_{ij}^+}{b_{ji}^-}$ $(\forall i\in I)$. Besides, according to Theorem 2, $A=(a_{ij})_{n\times n}$ is an additive consistent fuzzy preference relation, where $a_{ij}=0.5+\log_\alpha\frac{b_{ij}}{b_{ji}}(\forall i,j\in I)$. So there exists an additive consistent fuzzy preference relation $A=(a_{ij})_{n\times n}$, such that $a_{ij}^-\leqslant a_{ij}\leqslant a_{ij}^+(\forall i,j\in I)$. According to Definition 6, $\widetilde{A}$ is an additive consistent IFPR. □

For an additive consistent IFPR $\widetilde{A}$, through Eq. (13) with $\beta=81>1$, it can be changed to a multiplicative consistent IFPR $\widetilde{B}$,

$$\begin{bmatrix}\left[\frac{1}{2},\frac{1}{2}\right] & \left[\frac{3}{4}-\frac{1}{4}\log_3 7,\frac{1}{4}+\frac{1}{4}\log_3 2\right]\\ \left[\frac{3}{4}-\frac{1}{4}\log_3 2,\frac{1}{4}+\frac{1}{4}\log_3 7\right] & \left[\frac{1}{2},\frac{1}{2}\right]\end{bmatrix}=\widetilde{A}\overset{(13)}{\rightarrow}\widetilde{B}$$
$$=\begin{bmatrix}\left[\frac{1}{2},\frac{1}{2}\right] & \left[\frac{3}{10},\frac{2}{5}\right]\\ \left[\frac{3}{5},\frac{7}{10}\right] & \left[\frac{1}{2},\frac{1}{2}\right]\end{bmatrix}.$$

At the same time, through Eq. (14) with $\alpha=81>49/9=(\max\{1,2/3,7/3\})^2$, a multiplicative consistent IFPR $\widetilde{B}$ can be changed to an additive consistent IFPR $\widetilde{A}$,

$$\begin{bmatrix}\left[\frac{1}{2},\frac{1}{2}\right] & \left[\frac{3}{10},\frac{2}{5}\right]\\ \left[\frac{3}{5},\frac{7}{10}\right] & \left[\frac{1}{2},\frac{1}{2}\right]\end{bmatrix}=\widetilde{B}\overset{(14)}{\rightarrow}\widetilde{A}$$
$$=\begin{bmatrix}\left[\frac{1}{2},\frac{1}{2}\right] & \left[\frac{3}{4}-\frac{1}{4}\log_3 7,\frac{1}{4}+\frac{1}{4}\log_3 2\right]\\ \left[\frac{3}{4}-\frac{1}{4}\log_3 2,\frac{1}{4}+\frac{1}{4}\log_3 7\right] & \left[\frac{1}{2},\frac{1}{2}\right]\end{bmatrix}.$$

Based on Theorems 3 and 4, a multiplicative consistent IFPR can be changed into an additive consistent IFPR. And how to collect additive consistent information, transform back into multiplicative consistent information and derive interval weights will be introduced in Section 3.

3. Deriving interval weights from an IFPR

For an IFPR $\widetilde{B}=\left(\left[b_{ij}^-,b_{ij}^+\right]\right)_{n\times n}$

$$\widetilde{B}=\begin{bmatrix}[b_{11}^-,b_{11}^+] & [b_{12}^-,b_{12}^+] & \cdots & [b_{1n}^-,b_{1n}^+]\\ [b_{21}^-,b_{21}^+] & [b_{22}^-,b_{22}^+] & \cdots & [b_{2n}^-,b_{2n}^+]\\ \vdots & \vdots & \vdots & \vdots\\ [b_{n1}^-,b_{n1}^+] & [b_{n2}^-,b_{n2}^+] & \cdots & [b_{nn}^-,b_{nn}^+]\end{bmatrix},$$

to collect multiplicative consistent information and derive interval weights from the IFPR $\widetilde{B}=\left(\left[b_{ij}^-,b_{ij}^+\right]\right)_{n\times n}$, it can be changed to another IFPR $\widetilde{A}=\left(\left[a_{ij}^-,a_{ij}^+\right]\right)_{n\times n}$

$$\widetilde{A}=(\tilde{a}_{ij})_{n\times n}=\begin{bmatrix}[a_{11}^-,a_{11}^+] & [a_{12}^-,a_{12}^+] & \cdots & [a_{1n}^-,a_{1n}^+]\\ [a_{21}^-,a_{21}^+] & [a_{22}^-,a_{22}^+] & \cdots & [a_{2n}^-,a_{2n}^+]\\ \vdots & \vdots & \vdots & \vdots\\ [a_{n1}^-,a_{n1}^+] & [a_{n2}^-,a_{n2}^+] & \cdots & [a_{nn}^-,a_{nn}^+]\end{bmatrix}$$

by Eq. (14) with $\alpha>\left(\max_{i,j\in I}\frac{b_{ji}^+}{b_{ij}^-}\right)^2$.

3.1. Collecting additive consistent information from an additive consistent IFPR $\widetilde{A}$

If $\widetilde{B} = \left(\left[b_{ij}^-, b_{ij}^+\right]\right)_{n\times n}$ is a multiplicative consistent IFPR, according to Theorem 4, then $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$ is an additive consistent IFPR. Collecting additive consistent information from an additive consistent IFPR $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$ will be introduced.

Theorem 5. *If $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$ is an additive consistent IFPR, then for any $j \in I$, $\bigcap_{i\in I}\left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right)$ is not empty.*

Proof. Assume $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$ is an additive consistent IFPR, there exists an additive consistent FPR $A = (a_{ij})_{n\times n}$, such that

$$a_{ij}^- \leqslant a_{ij} \leqslant a_{ij}^+, \quad \forall i,j \in I. \tag{15}$$

Especially, when $j = 1$,

$$a_{i1}^- \leqslant a_{i1} \leqslant a_{i1}^+, \quad \forall i \in I. \tag{16}$$

For any $j \in I$,

$$a_{ij}^- - a_{i1}^+ \leqslant a_{ij} - a_{i1} \leqslant a_{ij}^+ - a_{i1}^-, \quad \forall i \in I. \tag{17}$$

Since $A = (a_{ij})_{n\times n}$ is an additive consistent FPR, so

$$a_{ij} = a_{i1} - a_{j1} + 0.5, \quad \forall i,j \in I. \tag{18}$$

Eq. (17) can be changed to

$$a_{ij}^- - a_{i1}^+ \leqslant -a_{j1} + 0.5 \leqslant a_{ij}^+ - a_{i1}^-, \quad \forall i \in I, \tag{19}$$

so

$$-a_{j1} + 0.5 \in \bigcap_{i\in I}\left[a_{ij}^- - a_{i1}^+, a_{ij}^+ - a_{i1}^-\right] = \bigcap_{i\in I}\left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right).$$

Therefore, $\bigcap_{i\in I}\left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right)$ is not empty. □

If $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$ is an additive consistent IFPR, let $\widetilde{T}_j = \bigcap_{i\in I}\left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right) = \left[T_j^-, T_j^+\right]$, then

$$\widetilde{T}_j \subseteq \left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right), \quad \forall i \in I.$$

On the one side, for all $i_0 \in I$, if

$$\widetilde{T}_j \subsetneqq \left(\left[a_{i_0j}^-, a_{i_0j}^+\right] - \left[a_{i_01}^-, a_{i_01}^+\right]\right),$$

then, there exists extra information, not additive consistent, concluded in $\left[a_{i_01}^-, a_{i_01}^+\right]$. To collect the additive consistent information, reset it by

$$\left[\bar{a}_{i_01}^-, \bar{a}_{i_01}^+\right] = \left[a_{i_01}^-, a_{i_01}^+\right] \bigcap \left(\left[a_{i_0j}^-, a_{i_0j}^+\right] - \left[T_j^-, T_j^+\right]\right).$$

Proposition 1. *If $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$ is an IFPR, then the intersection $\left[a_{i_01}^-, a_{i_01}^+\right] \bigcap \left(\left[a_{i_0j}^-, a_{i_0j}^+\right] - \left[T_j^-, T_j^+\right]\right)$ is not empty.*

Proof. Assume, on the contrary, the intersection $\left[a_{i_01}^-, a_{i_01}^+\right] \bigcap \left(\left[a_{i_0j}^-, a_{i_0j}^+\right] - \left[T_j^-, T_j^+\right]\right)$ is empty. It is to say $\left[a_{i_01}^-, a_{i_01}^+\right] \bigcap \left[a_{i_0j}^- - T_j^+, a_{i_0j}^+ - T_j^-\right]$ is empty. We now consider the following two cases.

Case 1: $a_{i_01}^- > a_{i_0j}^+ - T_j^-$, then $T_j^- > a_{i_0j}^+ - a_{i_01}^-$. But, according to $\left[T_j^-, T_j^+\right] = \bigcap_{i\in I}\left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right)$, $T_j^- \leqslant T_j^+ \leqslant \min_{i\in I} a_{ij}^+ - a_{i1}^- \leqslant a_{i_0j}^+ - a_{i_01}^-$.

Case 2: $a_{i_01}^+ < a_{i_0j}^- - T_j^+$, then $T_j^+ < a_{i_0j}^- - a_{i_01}^+$. But, according to $\left[T_j^-, T_j^+\right] = \bigcap_{i\in I}\left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right)$, $T_j^+ \geqslant T_j^- \geqslant \max_{i\in I} a_{ij}^- - a_{i1}^+ \geqslant a_{i_0j}^- - a_{i_01}^+$. Both two cases are contradiction, the intersection $\left[a_{i_01}^-, a_{i_01}^+\right] \bigcap \left(\left[a_{i_0j}^-, a_{i_0j}^+\right] - \left[T_j^-, T_j^+\right]\right)$ is not empty. □

On the other side, for all $i_0' \in I$, if

$$\widetilde{T}_j = \left(\left[a_{i_0'j}^-, a_{i_0'j}^+\right] - \left[a_{i_0'1}^-, a_{i_0'1}^+\right]\right),$$

then, $\left[a_{i_0'1}^-, a_{i_0'1}^+\right]$ is filled with additive consistent information. Don't need to adjust the first column.

Proposition 2. *Let $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$ is an IFPR, if $\widetilde{T}_j = \left(\left[a_{i_0'j}^-, a_{i_0'j}^+\right] - \left[a_{i_0'1}^-, a_{i_0'1}^+\right]\right)$, then $\left[a_{i_0'1}^-, a_{i_0'1}^+\right] \subseteq \left(\left[a_{i_0'j}^-, a_{i_0'j}^+\right] - \left[T_j^-, T_j^+\right]\right)$.*

Proof. Since $\widetilde{T}_j = \left[T_j^-, T_j^+\right] = \left(\left[a_{i_0'j}^-, a_{i_0'j}^+\right] - \left[a_{i_0'1}^-, a_{i_0'1}^+\right]\right)$, we have $T_j^- = a_{i_0'j}^- - a_{i_0'1}^+$ and $T_j^+ = a_{i_0'j}^+ - a_{i_0'1}^-$. Then $a_{i_0'1}^+ = a_{i_0'j}^- - T_j^- \leqslant a_{i_0'j}^+ - T_j^-$ and $a_{i_0'1}^- = a_{i_0'j}^+ - T_j^+ \geqslant a_{i_0'j}^- - T_j^+$. So $a_{i_0'j}^- - T_j^+ \leqslant a_{i_0'1}^- \leqslant a_{i_0'1}^+ \leqslant a_{i_0'j}^+ - T_j^-$, it is the same as $\left[a_{i_0'1}^-, a_{i_0'1}^+\right] \subseteq \left[a_{i_0'j}^- - T_j^+, a_{i_0'j}^+ - T_j^-\right]$. □

In this case, the first column can also be set as

$$\left[\bar{a}_{i_0'1}^-, \bar{a}_{i_0'1}^+\right] = \left[a_{i_0'1}^-, a_{i_0'1}^+\right] \bigcap \left(\left[a_{i_0'j}^-, a_{i_0'j}^+\right] - \left[T_j^-, T_j^+\right]\right).$$

In both sides above, set the new first column as follows,

$$\bar{a}_{i_1}^- = \max\left\{a_{i_1}^-, a_{i_j}^- - T_j^+\right\}, \quad \bar{a}_{i_1}^+ = \min\left\{a_{i_1}^+, a_{i_j}^+ - T_j^-\right\}, \quad \forall i \in I.$$

Based on the discussion, there is an algorithm to collect all the additive consistent information to the first column $[\bar{a}_{i1}^-, \bar{a}_{i1}^+]$ $(i \in I)$.

Algorithm 1

Input: An additive consistent interval fuzzy preference relation $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ and error ε.

Output: $([\bar{a}_{11}^-, \bar{a}_{11}^+], [\bar{a}_{21}^-, \bar{a}_{21}^+], \cdots, [\bar{a}_{n1}^-, \bar{a}_{n1}^+])^T$.

Step 1: Set $k = 1, j = 2$.

Step 2: Calculate $\tilde{t}_i = \left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+], \forall i \in I$, and let $\widetilde{T}_j = \cap_{i=1}^n \tilde{t}_i$.

Step 3: Note $\widetilde{T}_j = \left[T_j^-, T_j^+\right]$ and set

$$a_{i1}^- = a_{i1}^{kj-} = \max\left\{a_{i1}^-, a_{ij}^- - T_j^+\right\}, \quad a_{i1}^+ = a_{i1}^{kj+} = \min\left\{a_{i1}^+, a_{ij}^+ - T_j^-\right\},$$

$$a_{1i}^+ = 1 - a_{i1}^-, \quad a_{1i}^- = 1 - a_{i1}^+, \quad \forall i \in I.$$

Step 4: If $j < n$, set $j = j + 1$, go to Step 2.

Step 5: If $d_k = \frac{1}{2n}\sum_{i\in I}\left[|a_{i1}^{kn-} - a_{i1}^{(k-1)n-}| + |a_{i1}^{kn+} - a_{i1}^{(k-1)n+}|\right] < \varepsilon$, then let $[\bar{a}_{i1}^-, \bar{a}_{i1}^+] = [a_{i1}^-, a_{i1}^+] (i \in I)$and return $[\bar{a}_{i1}^-, \bar{a}_{i1}^+] (i \in I)$; Otherwise, set $k = k + 1, j = 2$, and go to Step 2.

3.2. Collecting additive consistent information from an IFPR $\widetilde{A}$ without additive consistency

If $\widetilde{B} = \left(\left[b_{ij}^-, b_{ij}^+\right]\right)_{n\times n}$ is not a multiplicative consistent IFPR, then $\widetilde{A} = \left(\left[a_{ij}^-, a_{ij}^+\right]\right)_{n\times n}$, achieved through Eq. (14) with $\alpha > \left(\max_{i,j\in I} \frac{b_{ji}^+}{b_{ij}^-}\right)^2$, is not an additive consistent IFPR. According to Theorem 5, the intersection $\bigcap_{i\in I}\left(\left[a_{ij}^-, a_{ij}^+\right] - [a_{i1}^-, a_{i1}^+]\right)$ may empty. If there exist $j_0 \in I$, the intersection $\bigcap_{i\in I}\left(\left[a^-_{ij_0}, a^+_{ij_0}\right] - \left[a^-_{i1}, a^+_{i1}\right]\right) = \bigcap_{i\in I}\left[a^-_{ij_0} - a^+_{i1}, a^+_{ij_0} - a^-_{i1}\right]$ is empty. It is to say that

$\max_{i\in I}\left(a^-_{ij_0} - a^+_{i1}\right) > \min_{i\in I}\left(a^+_{ij_0} - a^-_{i1}\right)$.

It means that the j_0th column doesn't contain additive consistent information. To adjust this column, the average of $\max_{i\in I}\left(a^-_{ij_0} - a^+_{i1}\right)$ and $\min_{i\in I}\left(a^+_{ij_0} - a^-_{i1}\right)$ is used to approach the intersection $\left[T^-_{j_0}, T^+_{j_0}\right]$. Let $T^-_{j_0} = T^+_{j_0} = \frac{1}{2}\left[\max_{i\in I}\left(a^-_{ij_0} - a^+_{i1}\right) + \min_{i\in I}\left(a^+_{ij_0} - a^-_{i1}\right)\right]$, and then the j_0th column can be adjusted by

$$a^-_{ij_0} = T^-_{j_0} + a^+_{i1}; a^+_{j_0 i} = 1 - a^-_{ij_0}, \quad \forall i \in \{i \in I : a^-_{ij_0} - a^+_{i1} > T^-_{j_0},\ i \neq j_0\};$$
$$a^+_{ij_0} = T^+_{j_0} + a^-_{i1}; a^-_{j_0 i} = 1 - a^+_{ij_0}, \quad \forall i \in \{i \in I : a^+_{ij_0} - a^-_{i1} < T^+_{j_0},\ i \neq j_0\}.$$

In this way, there is a new algorithm to collect additive consistent information from the inconsistent IFPR $\widetilde{A}$ to the first column $\left[\bar{a}^-_{i1}, \bar{a}^+_{i1}\right]$ $(i \in I)$.

Algorithm 2

Input: An interval fuzzy preference relation $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ and error ε.
Output: $\left(\left[\bar{a}^-_{11}, \bar{a}^+_{11}\right], \left[\bar{a}^-_{21}, \bar{a}^+_{21}\right], \ldots, \left[\bar{a}^-_{n1}, \bar{a}^+_{n1}\right]\right)^T$.
Step 1: Set $k = 1$, $j = 2$.
Step 2: Calculate $\tilde{t}_i = \left[a^-_{ij}, a^+_{ij}\right] - \left[a^-_{i1}, a^+_{i1}\right]$ $(i \in I)$, and let $\widetilde{T}_j = \cap^n_{i=1}\tilde{t}_i$.
Step 3: If $\widetilde{T}_j$ is empty, the interval fuzzy preference relation is inconsistent, set $T^-_j = T^+_j = \frac{1}{2}(\max_{i\in I} t^-_i + \min_{i\in I} t^+_i)$ and set

$$a^-_{ij} = T^-_j + a^+_{i1}; a^+_{ji} = 1 - a^-_{ij}, \quad \forall i \in \left\{i \in I : a^-_{ij} - a^+_{i1} > T^-_j, i \neq j_0\right\};$$
$$a^+_{ij} = T^+_j + a^-_{i1}; a^-_{ji} = 1 - a^+_{ij}, \quad \forall i \in \left\{i \in I : a^+_{ij} - a^-_{i1} < T^+_j, i \neq j_0\right\}.$$

Otherwise, note $\widetilde{T}_j = [T^-_j, T^+_j]$ and set

$$a^-_{i1} = a^{kj-}_{i1} = \max\left\{a^-_{i1}, a^-_{ij} - T^+_j\right\}, \quad a^+_{i1} = a^{kj+}_{i1} = \min\left\{a^+_{i1}, a^+_{ij} - T^-_j\right\},$$
$$a^+_{1i} = 1 - a^-_{i1}, a^-_{1i} = 1 - a^+_{i1}, \quad \forall i \in I.$$

Step 4: If $j < n$, set $j = j + 1$, go to Step 2.
Step 5: If $d_k = \frac{1}{2n}\sum_{i\in I}\left[|a^{kn-}_{i1} - a^{(k-1)n-}_{i1}| + |a^{kn+}_{i1} - a^{(k-1)n+}_{i1}|\right] < \varepsilon$, then let $\left[\bar{a}^-_{i1}, \bar{a}^+_{i1}\right] = \left[a^-_{i1}, a^+_{i1}\right]$ $(i \in I)$ and return $\left[\bar{a}^-_{i1}, \bar{a}^+_{i1}\right] (i \in I)$; Otherwise, set $k = k + 1$, $j = 2$, and go to Step 2.

According to Theorem 5, if $\widetilde{A}$ is an additive consistent IFPR, then $\widetilde{T}_j$ is not empty. In this case, the Step 3 in Algorithm 2 is the same as in Algorithm 1, then Algorithm 2 is the same as Algorithm 2. In another word, no matter IFPR $\widetilde{A}$ is additive consistent or not, Algorithm 2 can be used to collect all the additive consistent information to $\left[\bar{a}^-_{i1}, \bar{a}^+_{i1}\right]$ $(i \in I)$

$$\left(\left[\bar{a}^-_{11}, \bar{a}^+_{11}\right], \left[\bar{a}^-_{21}, \bar{a}^+_{21}\right], \ldots, \left[\bar{a}^-_{n1}, \bar{a}^+_{n1}\right]\right)^T.$$

Then, it can be transformed into $\left[\bar{b}^-_{1j}, \bar{b}^+_{1j}\right]$ $(j \in I)$ through Eq. (13) with $\beta = \alpha > 1$,

$$\left(\left[\bar{b}^-_{11}, \bar{b}^+_{11}\right], \left[\bar{b}^-_{12}, \bar{b}^+_{12}\right], \ldots, \left[\bar{b}^-_{1n}, \bar{b}^+_{1n}\right]\right),$$

which contains all the multiplicative consistent information in an IFPR $\widetilde{B}$.

The following part is to calculate interval weights from the collected multiplicative consistent information $\left[\bar{b}^-_{1j}, \bar{b}^+_{1j}\right]$ $(j \in I)$.

3.3. An algorithm to derive interval weights from an IFPR $\widetilde{B}$

Theorem 6. [20] *Let $\widetilde{B} = \left(\left[b^-_{ij}, b^+_{ij}\right]\right)_{n\times n}$ be a multiplicative consistent IFPR, and note $\bar{b}^-_{1j} = \min_\Phi b_{1j}$ and $\bar{b}^+_{1j} = \max_\Phi b_{1j}$, $\forall j \in I$, then*

$$\min_\Omega w_1 = \left(\sum_{j\in I}\frac{1 - \bar{b}^-_{1j}}{\bar{b}^-_{1j}}\right)^{-1}, \quad \max_\Omega w_1 = \left(\sum_{j\in I}\frac{1 - \bar{b}^+_{1j}}{\bar{b}^+_{1j}}\right)^{-1}, \quad i \in I, \tag{20}$$

$w = (w_1, w_2, \ldots, w_n) \in \Omega$,

where $\Phi = \{(b_{ij})_{n\times n} \in R^{n\times n} | b^-_{ij} \leqslant b_{ij} \leqslant b^+_{ij}, b_{ij}b_{jk}b_{ki} = b_{ik}b_{kj}b_{ji},\ \forall i, j, k \in I\}$ *and* $\Omega = \{(w_1, w_2, \ldots, w_n) \in R^n | w_i \geqslant 0,\ \sum^n_{i=1} w_i = 1, b^-_{ij} \leqslant \frac{w_i}{w_i + w_j} \leqslant b^+_{ij},\ \forall i, j \in I\}$.

Based on Theorem 6 and Algorithm 2 above, there is an algorithm to derive all the interval weights from an IFPR $\widetilde{B} = \left(\left[b^-_{ij}, b^+_{ij}\right]\right)_{n\times n}$, no matter the IFPR is multiplicative consistent or not.

Algorithm 3

Input: A multiplicative consistent IFPR $\widetilde{B} = \left(\left[b^-_{ij}, b^+_{ij}\right]\right)_{n\times n}$ and error ε.
Output: $\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n$.
Step 1: Set $m = 1, \alpha = \beta > \left(\max_{i,j\in I}\frac{b^+_{ji}}{b^-_{ij}}\right)^2$.
Step 2: Transform the multiplicative consistent IFPR $\widetilde{B} = \left(\left[b^-_{ij}, b^+_{ij}\right]\right)_{n\times n}$ into an additive consistent IFPR $\widetilde{A} = \left(\left[a^-_{ij}, a^+_{ij}\right]\right)_{n\times n}$ through Eq. (14).
Step 3: Through Algorithm 2, input the additive consistent IFPR $\widetilde{A} = \left(\left[a^-_{ij}, a^+_{ij}\right]\right)_{n\times n}$ and error ε, then all additive consistent information $\left[\bar{a}^-_{i1}, \bar{a}^+_{i1}\right](i \in I)$ will be outputted.
Step 4: Transform $\left[\bar{a}^-_{i1}, \bar{a}^+_{i1}\right](i \in I)$ into $\left[\bar{b}^-_{1j}, \bar{b}^+_{1j}\right](j \in I)$ through Eq. (13).
Step 5: Derive the interval weight $\tilde{w}_m$ by

$$\tilde{w}_m = \left[\left(\sum_{j\in I}\frac{1 - \bar{b}^-_{1j}}{\bar{b}^-_{1j}}\right)^{-1}, \left(\sum_{j\in I}\frac{1 - \bar{b}^+_{1j}}{\bar{b}^+_{1j}}\right)^{-1}\right], i \in I.$$

Step 6: If $m < n$, set $m = m + 1$, exchange both the mth row and column with the first in $\widetilde{B} = \left(\left[b^-_{ij}, b^+_{ij}\right]\right)_{n\times n}$ and go to Step 2; Otherwise, return $\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n$.

All weights which output from Algorithm 3 are interval numbers. To rank interval weights, a straightforward possibility-degree formula introduced by Xu and Da [24] is used to compare two interval weights.

Definition 8. [24] Let $\tilde{w}_i = \left[w^-_i, w^+_i\right]$ and $\tilde{w}_j = \left[w^-_j, w^+_j\right]$ be any two interval weights, where $0 \leqslant w^-_i \leqslant w^+_i \leqslant 1$ and $0 \leqslant w^-_j \leqslant w^+_j \leqslant 1$, then the degree of possibility of $\tilde{w}_i \geqslant \tilde{w}_j$ is defined as

$$p(\tilde{w}_i \geqslant \tilde{w}_j) = \max\left\{1 - \max\left\{\frac{w^+_j - w^-_i}{w^+_i - w^-_i + w^+_j - w^-_j}, 0\right\}, 0\right\}. \tag{21}$$

That is, $\tilde{w}_i$ is superior to $\tilde{w}_j$ to degree of $p(\tilde{w}_i \geqslant \tilde{w}_j)$, denoted by $\tilde{w}_i \overset{p(\tilde{w}_i \geqslant \tilde{w}_j)}{\succ} \tilde{w}_j$. Especially, $p(\tilde{w}_i \geqslant \tilde{w}_j) > 0.5$ indicates that $\tilde{w}_i$ is superior to $\tilde{w}_j$ to degree of $p(\tilde{w}_i \geqslant \tilde{w}_j)$; $p(\tilde{w}_i \geqslant \tilde{w}_j) = 0.5$ indicates that $\tilde{w}_i$ is the same as $\tilde{w}_j$; $p(\tilde{w}_i \geqslant \tilde{w}_j) < 0.5$ indicates that $\tilde{w}_j$ is superior to $\tilde{w}_i$ to degree of $1 - p(\tilde{w}_i \geqslant \tilde{w}_j)$.

4. Numerical examples

Example 1. In SWOT analysis [25], decision makers compare each pair of elements in {*Strengths*, *Weaknesses*, *Opportunities*, *Threats*} and provide their preferences, respectively. All preferences are collected in

$$\widetilde{B}=\begin{bmatrix}[0.5,0.5] & [0.36,0.66] & [0.26,0.45] & [0.57,0.72]\\ [0.34,0.63] & [0.5,0.5] & [0.32,0.52] & [0.55,0.77]\\ [0.55,0.74] & [0.48,0.68] & [0.5,0.5] & [0.66,0.83]\\ [0.28,0.43] & [0.23,0.45] & [0.17,0.34] & [0.5,0.5]\end{bmatrix}.$$

The following steps can help to derive interval weights from $\widetilde{B}$ and rank the four elements of *Strengths* (*S*), *Weaknesses* (*W*), *Opportunities* (*O*) and *Threats* (*T*):

Step 1: Calculate $\left(\max_{i,j\in I}\frac{b_{ji}^{+}}{b_{ij}^{-}}\right)^2\approx 4.88^2$ and let $\alpha=\beta=25$.
Step 2: Then the multiplicative consistent IFPR $\widetilde{B}$ can be changed to an additive consistent IFPR $\widetilde{A}$ through Eq. (14)

$$\widetilde{A}=\begin{bmatrix}[0.5000,0.5000] & [0.3261,0.7061] & [0.1751,0.4377] & [0.5876,0.7934]\\ [0.2939,0.6739] & [0.5000,0.5000] & [0.2658,0.5249] & [0.5623,0.8754]\\ [0.5623,0.8249] & [0.4751,0.7342] & [0.5000,0.5000] & [0.7061,0.9926]\\ [0.2066,0.4124] & [0.1246,0.4377] & [0.0074,0.2939] & [0.5000,0.5000]\end{bmatrix}.$$

Step 3: Through Algorithm 2 with $\varepsilon=0.001$, input $\widetilde{A}$, then output the additive consistent information $[\bar{a}_{i1}^{-},\bar{a}_{i1}^{+}]$ $(i\in I)$ that $([0.5,0.5][0.3282,0.6739][0.5623,0.8249][0.2066,0.4124])^T$.
Step 4: Through Eq. (13), transform all the additive consistent information $[\bar{a}_{i1}^{-},\bar{a}_{i1}^{+}](i\in I)$ into $\left[\bar{b}_{1j}^{-},\bar{b}_{1j}^{+}\right]$ $(j\in I)$ that ([0.5,0.5] [0.3636,0.6349][0.26,0.45][0.57,0.72]).
Step 5: Derive the interval weight $\tilde{w}_1=[0.1575,0.3138]$ by Eq. (20).
Step 6: Exchange both the *m*th row and column with the first in $\widetilde{B}=\left(\left[b_{ij}^{-},b_{ij}^{+}\right]\right)_{n\times n}$, go to Step 2 and derive all the interval weights that $\tilde{w}_2=[0.1760,0.3580]$, $\tilde{w}_3=[0.2927,0.4934]$, $\tilde{w}_4=[0.0847,0.1822]$.

Then, the possibility-degree formula can be used to compare each pair of $\tilde{w}_i$ and $\tilde{w}_j$ $(i,j=1,2,3,4)$ by Eq. (21), and construct the following fuzzy preference relation:

$$P=\begin{bmatrix}0.5 & 0.4074 & 0.0593 & 0.9026\\ 0.5926 & 0.5 & 0.1707 & 0.9779\\ 0.9407 & 0.8293 & 0.5 & 1\\ 0.0974 & 0.0221 & 0 & 0.5\end{bmatrix}.$$

Summing all the elements of each row of *P*, we get:
$p_1=1.8692$, $p_2=2.2412$, $p_3=3.2700$, $p_4=0.6195$. Then,

$$w_3\overset{83\%}{\succ}w_2\overset{59\%}{\succ}w_1\overset{90\%}{\succ}w_4.$$

Therefore, *Strengths* (*S*), *Weaknesses* (*W*), *Opportunities* (*O*), *Threats* (*T*) can be ranked as follows:

Opportunities $\succ$ *Weaknesses* $\succ$ *Strengths* $\succ$ *Threats*.

Gao and Peng [25] utilized the UOWA operator which had been proposed by Xu and Da [24] to derive the priority weights using the lower limits of interval numbers. It is rough that only the lower limits were taken into account. Most information in interval numbers may be lost. With their method, the priority weights of $\widetilde{B}$ is that $w_1=0.264$, $w_2=0.269$, $w_3=0.284$, $w_4=0.183$. It's obviously that almost all w_i are contained in $\tilde{w}_i$ $(i\in I)$.

Example 2. Consider a fuzzy multiple criteria decision-making problem with a finite set of 5 criteria, let $X=\{x_1,x_2,x_3,x_4,x_5\}$ be the set of criteria and let $I=\{1,2,3,4,5\}$ be the set of index. A decision maker compares each pair of criteria in *X*, and provides his/her preference degree $\tilde{b}_{ij}=\left[b_{ij}^{-},b_{ij}^{+}\right]$ $(i,j\in I)$ of the criterion x_i over x_j. All these preference degrees $\tilde{b}_{ij}$ $(i,j\in I)$ compose a multiplicative consistent IFPR $\widetilde{B}=(\tilde{b}_{ij})_{n\times n}$ [20]

$$\widetilde{B}=\begin{bmatrix}[0.5,0.5] & [0.5,0.6] & [0.2,0.5] & [0.3,0.7] & [0.3,0.5]\\ [0,4,0.5] & [0.5,0.5] & [0.3,0.6] & [0.6,0.8] & [0.2,0.4]\\ [0.5,0.8] & [0.4,0.7] & [0.5,0.5] & [0.7,0.8] & [0.4,0.5]\\ [0.3,0.7] & [0.2,0.4] & [0.2,0.3] & [0.5,0.5] & [0.1,0.4]\\ [0.5,0.7] & [0.6,0.8] & [0.5,0.6] & [0.6,0.9] & [0.5,0.5]\end{bmatrix}.$$

The following steps can help to derive interval weights and rank x_1,x_2,x_3,x_4,x_5:

Step 1: Calculate $\left(\max_{i,j\in I}\frac{b_{ji}^{+}}{b_{ij}^{-}}\right)^2=81$ and let $\alpha=\beta=90$.
Step 2: Then the multiplicative consistent IFPR $\widetilde{B}$ can be changed to an additive consistent IFPR $\widetilde{A}$ through Eq. (14)

$$\widetilde{A}=\begin{bmatrix}[0.5000,0.5000] & [0.5000,0.5901] & [0.1919,0.5000] & [0.3117,0.6883] & [0.3117,0.5000]\\ [0.4099,0.5000] & [0.5000,0.5000] & [0.3117,0.5901] & [0.5901,0.8081] & [0.1919,0.4099]\\ [0.5000,0.8081] & [0.4099,0.6883] & [0.5000,0.5000] & [0.6883,0.8081] & [0.4099,0.5000]\\ [0.3117,0.6883] & [0.1919,0.4099] & [0.1919,0.3117] & [0.5000,0.5000] & [0.0117,0.4099]\\ [0.5000,0.6883] & [0.5901,0.8081] & [0.5000,0.5901] & [0.5901,0.9883] & [0.5000,0.5000]\end{bmatrix}.$$

Step 3: Through Algorithm 2 with $\varepsilon=0.001$, input $\widetilde{A}$, then output the additive consistent information $[\bar{a}_{i1}^{-},\bar{a}_{i1}^{+}](i\in I)$ that $([0.5,0.5][0.4099,0.5][0.5,0.6883][0.3117,0.4099][0.5,0.6883])^T$.
Step 4: Through Eq. (13), transform all the additive consistent information $[\bar{a}_{i1}^{-},\bar{a}_{i1}^{+}]$ $(i\in I)$ into $\left[\bar{b}_{1j}^{-},\bar{b}_{1j}^{+}\right]$ $(j\in I)$ that ([0.5,0.5] [0.5,0.6][0.3,0.5][0.6,0.7][0.3,0.5]).
Step 5: Derive the interval weight $\tilde{w}_1=[0.1364,0.2442]$ by Eq. (20).
Step 6: Exchange both the *m*th row and column with the first in $\widetilde{B}=\left(\left[b_{ij}^{-},b_{ij}^{+}\right]\right)_{n\times n}$, go to Step 2 and derive all the interval weights that $\tilde{w}_2=[0.1111,0.2029]$, $\tilde{w}_3=[0.2029,0.3218]$, $\tilde{w}_4=[0.0662,0.1154]$, $\tilde{w}_5=[0.2442,0.3899]$.

Then, the possibility-degree formula can be used to compare each pair of $\tilde{w}_i$ and $\tilde{w}_j$ $(i,j = 1,2,3,4)$ by Eq. (21), and construct the following fuzzy preference relation:

$$P = \begin{bmatrix} 0.5 & 0.6668 & 0.1822 & 1 & 0 \\ 0.3332 & 0.5 & 0 & 0.9695 & 0 \\ 0.8178 & 1 & 0.5 & 1 & 0.2933 \\ 0 & 0.0305 & 0 & 0.5 & 0 \\ 1 & 1 & 0.7067 & 1 & 0.5 \end{bmatrix}.$$

Summing all the elements of each row of P, we get:

$$p_1 = 2.3490, \quad p_2 = 1.8027, \quad p_3 = 3.6111, \quad p_4 = 0.5305, \quad p_5 = 4.2067.$$

Then,

$$\tilde{w}_5 \overset{71\%}{\succ} \tilde{w}_3 \overset{82\%}{\succ} \tilde{w}_1 \overset{67\%}{\succ} \tilde{w}_2 \overset{97\%}{\succ} \tilde{w}_4.$$

Therefore, x_1, x_2, x_3, x_4 can be ranked as follows:

$$x_5 \succ x_3 \succ x_1 \succ x_2 \succ x_4.$$

Based on the interval multiplicative transitivity, Genc et al. [20] used an estimated interval fuzzy preference relation to replace the original to derive interval weights. However, all processes were based on repeatedly solving mathematical models, which is complex. With their method, all the derived interval weights of $\widetilde{B}$ are the same as the results above.

5. Conclusions

In this paper, based on the exchanges of an additive consistent IFPR and a multiplicative consistent IFPR, we have developed a method to derive interval weights from both multiplicative consistent IFPR and an inconsistent one. Firstly, Some exchanges between an additive consistent FPR and a multiplicative consistent FPR have been established. Then, these exchanges have been extended to IFPR and relations between an additive consistent IFPR and a multiplicative consistent IFPR have been established. Secondly, a multiplicative consistent IFPR has been changed into an additive consistent IFPR. Thirdly, all the additive consistent information has been collected and changed back into multiplicative consistent information in IFPR. Fourthly, the interval weights have been generated from the collected multiplicative consistent information. Finally, two numerical examples are given to illustrate the new method.

For an interval fuzzy preference relation, both multiplicative consistent relation and additive consistent relation can be used to derive the weights. However, what's the relationship of the two kinds of weights and which is more reasonable are unknown. Both of them need further research.

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